

Chapter 2

Simple operations and properties of sequences

2.1 Simple operations on signals

In analyzing discrete-time systems, operations on sequences occur frequently. Some operations are discussed below.

2.1.1 Sequence addition:

Let $\{x[n]\}$ and $\{y[n]\}$ be two sequences. The sequence addition is defined as term by term addition. Let $\{z[n]\}$ be the resulting sequence

$$\{z[n]\} = \{x[n]\} + \{y[n]\}$$

where each term $z[n] = x[n] + y[n]$

We will use the following notation

$$\{x[n]\} + \{y[n]\} = \{x[n] + y[n]\}$$

2.1.2 Scalar multiplication:

Let a be a scalar. We will take a to be real if we consider only the real valued signals, and take a to be a complex number if we are considering complex valued sequence. Unless otherwise stated we will consider complex valued sequences. Let the resulting sequence be denoted by $w[n]$

$$\{w[n]\} = ax[n]$$

is defined by

$$w[n] = ax[n],$$

each term is multiplied by a

We will use the notation $aw[n] = aw[n]$

Note: If we take the set of sequences and define these two operators as addition and scalar multiplication they satisfy all the properties of a linear vector space.

2.1.3 Sequence multiplication:

Let $\{x[n]\}$ and $\{y[n]\}$ be two sequences, and $\{z[n]\}$ be resulting sequence

$$\{z[n]\} = \{x[n]\}\{y[n]\}$$

where $z[n] = x[n]y[n]$

The notation used for this will be $\{x[n]\}\{y[n]\} = \{x[n]y[n]\}$

Now we consider some operations based on independent variable n .

2.1.4 Shifting

This is also known as translation. Let us shift a sequence $\{x[n]\}$ by n_0 units, and the resulting sequence by $\{y[n]\}$

$$\{y[n]\} = z^{-n_0}(\{x[n]\})$$

where $z^{-n_0}()$ is the operation of shifting the sequence right by n_0 unit. The terms are defined by $y[n] = x[n - n_0]$. We will use short notation $\{x[n - n_0]\}$ to denote shift by n_0 .

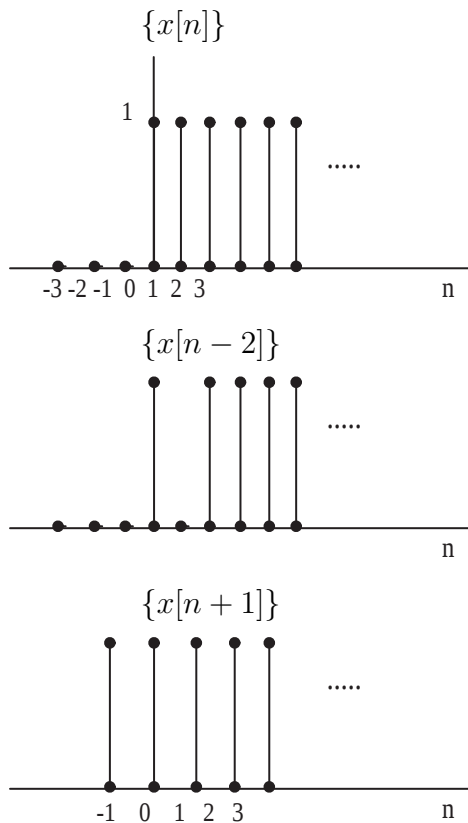
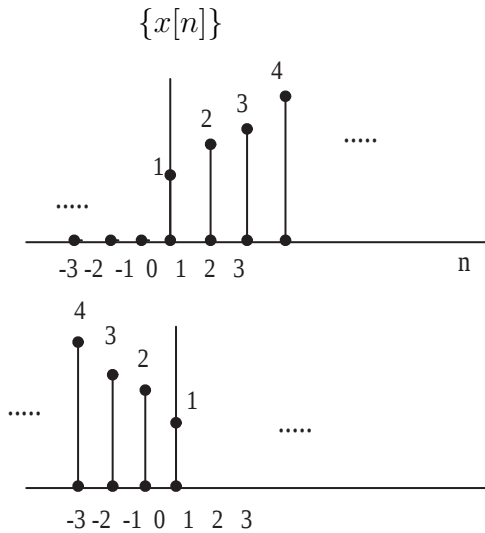


Figure above show some examples of shifting. A negative value of n_0 means shift towards right.

2.1.5 Reflection:

Let $\{x[n]\}$ be the original sequence, and $\{y[n]\}$ be reflected sequence, then $y[n]$ is defined by

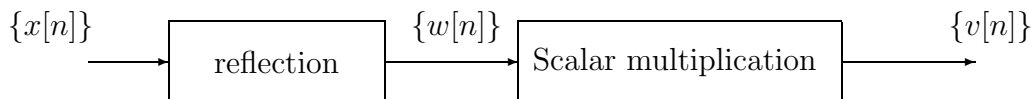
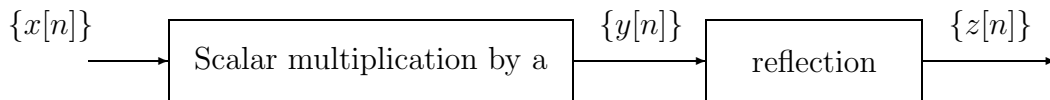
$$y[n] = x[-n]$$



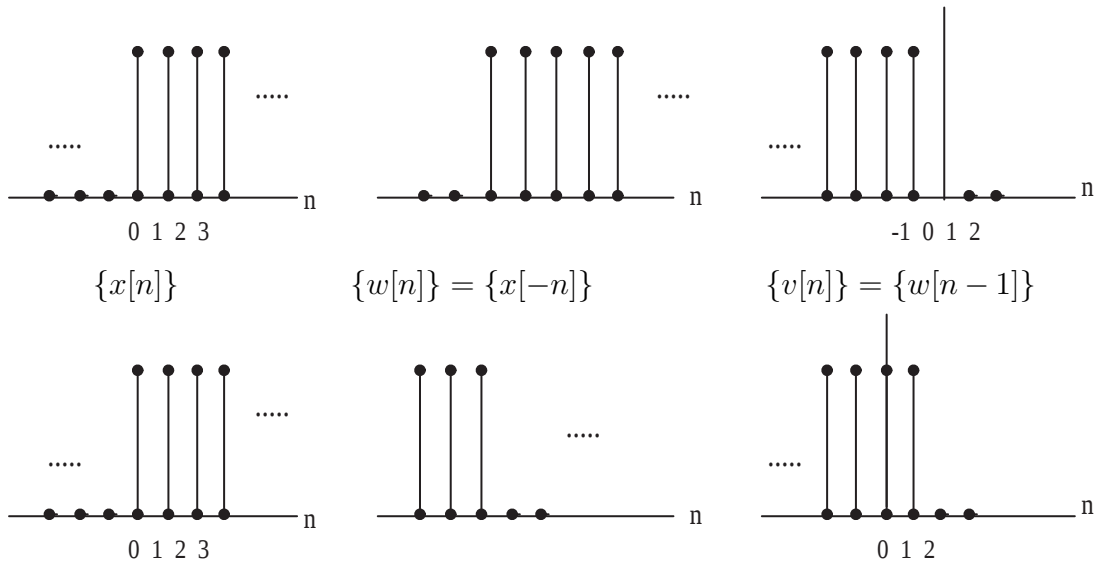
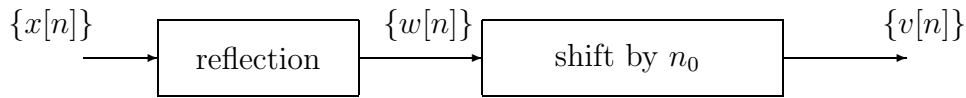
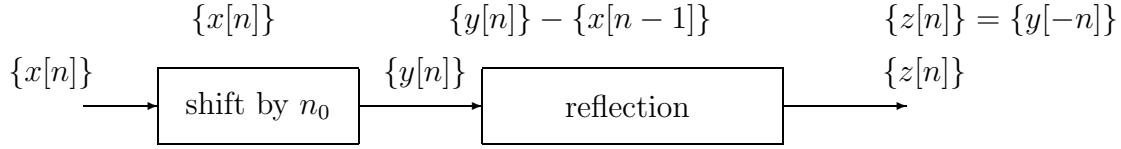
We will denote this by $\{x[n]\}$.

When we have complex valued signals, sometimes we reflect and do the complex conjugation, ie, $y[n]$ is defined by $y[n] = x^*[-n]$, where $*$ denotes complex conjugation. This sequence will be denoted by $\{x^*[-n]\}$.

We will learn about more complex operations later on. Some of these operations commute, ie. if we apply two operations we can interchange their order and some do not commute. For example scalar multiplication and reflection commute.



Then $v[n] = z[n]$ for all n . Shifting and scaling do not commute.



We can combine many of these operations in one step, for example $\{y[n]\}$ may be defined as $y[n] = 2x[3-n]$.

2.2 Some properties of signals:

2.2.1 Energy of a Signal:

The total energy of a signal $\{x[n]\}$ is defined by

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

A signal is referred to as an energy signal, if and only if the total energy of the signal E_x is finite.

2.2.2 Power of a signal:

If $\{x[n]\}$ is a signal whose energy is not finite, we define power of the signal as

$$P_x = \lim_{N \rightarrow \infty} \frac{1}{(2N+1)} \sum_{n=-N}^N |x[n]|^2$$

A signal is referred to as a power signal if the power P_x satisfies the condition

$$0 < P_x < \infty$$

An energy signal has a zero power and a power signal has infinite energy. There are signals which are neither energy signals nor power signals. For example $\{x[n]\}$ defined by $x[n] = n$ does not have finite power or energy.

2.2.3 Periodic Signals:

An important class of signals that we encounter frequently is the class of periodic signals. We say that a signal $\{x[n]\}$ is periodic period N , where N is a positive integer, if the signal is unchanged by the time shift of N ie.,

$$\{x[n]\} = \{x[n+N]\}$$

or $x[n] = x[n+N]$ for all n .

Since $\{x[n]\}$ is same as $\{x[n+N]\}$, it is also periodic so we get

$$\{x[n]\} = \{x[n+N]\} = \{x[n+N+N]\} = \{x[n+2N]\}$$

Generalizing this we get $\{x[n]\} = \{x[n+kN]\}$, where k is a positive integer. From this we see that $\{x[n]\}$ is periodic with $2N, 3N, \dots$. The fundamental period N_0 is the smallest positive value N for which the signal is periodic.

The signal illustrated below is periodic with fundamental period $N_0 = 4$

$\{x[n]\}$ By change of variable we can write $\{x[n]\} = \{x[n+N]\}$ as $\{x[m-N]\} = \{x[m]\}$ and thenas before, we see that

$$\{x[n]\} = \{x[n+kN]\},$$

for all integer values of k , positive, negative or zero. By definition, period of a signal is always a positive integer n .

Except for a all zero signal all periodic signals have infinite energy. They may

have finite power. Let $\{x[n]\}$ be periodic with period N , then the power P_x is given by

$$\begin{aligned}
P_x &= \lim_{M \rightarrow \infty} \frac{1}{2M+1} \sum_{n=-M}^M |x[n]|^2 \\
&= \lim_{M \rightarrow \infty} \frac{1}{2M+1} \left[\sum_{n=0}^{N-1} |x[n]|^2 + \sum_{n=N}^{2N-1} |x[n]|^2 + \dots \right. \\
&\quad + \sum_{n=(k-1)N}^{kN-1} |x[n]|^2 + \sum_{n=kN}^M |x[n]|^2 + \sum_{n=-N}^{-1} |x[n]|^2 + \dots \\
&\quad \left. + \sum_{n=-kN}^{-(k-1)N-1} |x[n]|^2 + \sum_{n=-M}^{-kN-1} |x[n]|^2 \right]
\end{aligned}$$

where k is largest integer such that $kN - 1 \leq M$. Since the signal is periodic, sum over one period will be same for all terms. We see that k is approximately equal to M/N (it is integer part of this) and for large M we get $2M/N$ terms and limit $2M/(2M+1)$ as M goes to infinite is one we get

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

2.2.4 Even and odd signals:

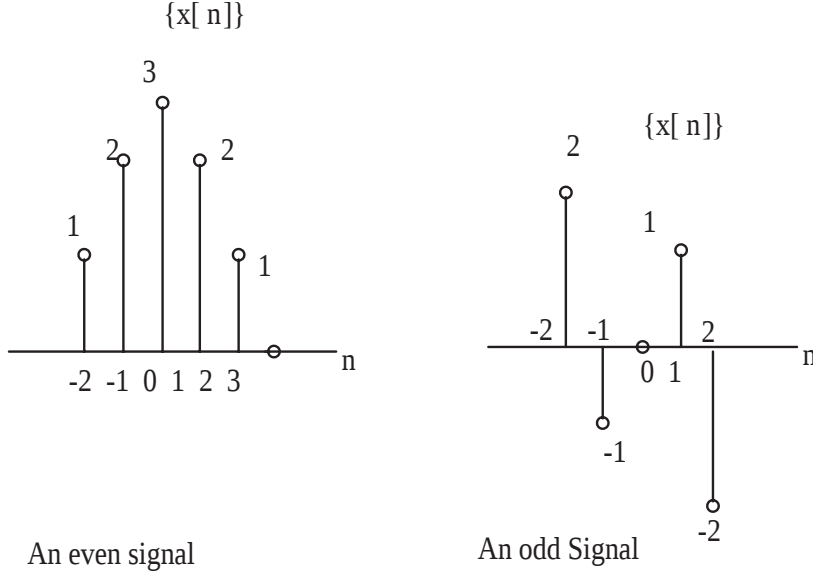
A real valued signal $\{x[n]\}$ is referred as an even signal if it is identical to its time reversed counterpart ie, if

$$\{x[n]\} = \{x[-n]\}$$

A real signal is referred to as an odd signal if

$$\{x[n]\} = \{-x[-n]\}$$

An odd signal has value 0 at $n = 0$ as $x[0] = -x[n] = -x[0]$



as a sum of an even signal and an odd signal. Consider the signals

$$Ev(\{x[n]\}) = \{x_e[n]\} = \{1/2(x[n] + x[-n])\}$$

and

$$Od(\{x[n]\}) = \{x_o[n]\} = \{1/2(x[n] - x[-n])\}$$

We can see easily that

$$\{x[n]\} = \{x_e[n]\} + \{x_o[n]\}$$

The signal $\{x_e[n]\}$ is called the even part of $\{x[n]\}$. We can verify very easily that $\{x_e[n]\}$ is an even signal. Similarly, $\{x_o[n]\}$ is called the odd part of $\{x[n]\}$ and is an odd signal. When we have complex valued signals we use a slightly different terminology. A complex valued signal $\{x[n]\}$ is referred to as a conjugate symmetric signal if

$$\{x[n]\} = \{x^*[-n]\}$$

where x^* refers to the complex conjugate of x . Here we do reflection and complex conjugation. If $\{x[n]\}$ is real valued this is same as an even signal. A complex signal $\{x[n]\}$ is referred to as a conjugate antisymmetric signal if

$$\{x[n]\} = \{-x^*[-n]\}$$

We can express any complex valued signal as sum conjugate symmetric and conjugate antisymmetric signals. We use notation similar to above

$$Ev(\{x[n]\}) = \{x_e[n]\} = \{1/2(x[n] + x^*[-n])\}$$

and

$$Od(\{x[n]\}) = \{x_o[n]\} = \{1/2(x[n] - x^*[-n])\}$$

then

$$\{x[n]\} = \{x_e[n]\} + \{x_o[n]\}$$

We can see easily that $\{x_e[n]\}$ is conjugate symmetric signal and $\{x_o[n]\}$ is conjugate antisymmetric signal. These definitions reduce to even and odd signals in case signals takes only real values.

2.2.5 Periodicity properties of sinusoidal signals

Let us consider the signal $\{x[n]\} = \{\cos w_0 n\}$. We see that if we replace w_0 by $(w_0 + 2\pi)$ we get the same signal. In fact the signal with frequency $w_0 \pm 2\pi, w_0 \pm 4\pi$ and so on. This situation is quite different from continuous time signal $\{\cos w_0 t, -\infty < t < \infty\}$ where each frequency is different. Thus in discrete time we need to consider frequency interval of length 2π only. As we increase w_0 to π signal oscillates more and more rapidly. But if we further increase frequency from π to 2π the rate of oscillations decreases. This can be seen easily by plotting signal $\cos w_0 n$ for several values of w_0 .

The signal $\{\cos w_0 n\}$ is not periodic for every value of w_0 . For the signal to be periodic with period $N > 0$, we should have

$$\{\cos w_0 n\} = \{\cos w_0 (n + N)\}$$

that is $w_0 N$ should be some multiple of 2π .

$$w_0 N = 2\pi m$$

or

$$\frac{w_0}{2\pi} = \frac{m}{N}$$

Thus signal $\{\cos w_0 n\}$ is periodic if and only if $\frac{w_0}{2\pi}$ is a rational number.

Above observations also hold for complex exponential signal $\{x[n]\} = \{e^{jw_0 n}\}$